Cost of Not Arbitrarily Splitting in Routing

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Outline

Problem Formulation

Bounding the Performance Gap

Routing Optimization with Split Ratio Granularity Constraints
Introduction

- Practical restrictions can prevent an optimized routing solution to be fully realized.
- Routers can put additional restrictions on routing solutions such as:
  - At most $W$ paths are allowed for each source-destination pair (Path cardinality constraints).
  - There is a minimum granularity for the split ratio (Split ratio granularity constraints).
Introduction

- Multipath routing achieves the best performance, but is hard to implement. Single-path routing is opposite.
- When the split ratio granularity becomes larger, routing performance decreases but implementation overhead also decreases.
Two Basic Questions

- How to estimate the performance gap between the multipath routing and routing with split ratio granularity constraints?
- How to find a good approximate solution to the split ratio granularity problem?
Notation

- Number of links $L = 5$. Number of users $N = 2$. Number of paths each user has $K^1 = K^2 = 2$.
- The paths of user $i$ are represented by

\[ R^1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad R^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

Put $R^i_{lk} = 1$ if path $k$ of user $i$ passes through link $l$.
Notation

- Link capacity $c = (c_1, \ldots, c_5)^T$.
- The $k$th entry $x^i_k$ of vector $x^i$ is the sending rate of user $i$ on path $k$.
- The utility $U^i(\cdot)$ of user $i$ is a function of its total transmission rate $\|x^i\|_1$.
- We focus on two common utility functions:
  - Linear utility $U^i(s) = s$.
  - Logarithmic utility $U^i(s) = \log s$. 

![Diagram showing network with users 1 and 2 and paths between nodes A, B, C, and D]
The network utility maximization (NUM) problem with multipath routing:

$$\max \sum_{i=1}^{N} U^i \left( \|x^i\|_1 \right)$$

s.t. $$\sum_{i=1}^{N} R^i x^i \leq c,$$

$$x^i \in \mathcal{I}_Ki, \quad \forall i = 1, \ldots, N.$$

- Here $$\mathcal{I}_K = \{x \in \mathbb{R}^K | 0 \leq x_k \leq \|c\|_{\infty}, k = 1, \ldots, K\}$$, while $$\|c\|_{\infty}$$ is the maximum capacity among all links in the network.
- Without loss of generality, assume $$\|c\|_{\infty} = 1$$.
- Replace $$\mathcal{I}_K$$ by a smaller set to introduce split ratio granularity constraints.
Split Ratio Granularity Constraints

- Suppose the split ratio of each user needs to be a multiple of $1/p$, where $p$ is a given integer.
- Choose the set

$$S_K = \{0\} \cup \left\{ x \in I_K \mid x \neq 0, \frac{px_k}{\|x\|_1} \in \mathbb{Z}, k = 1, \ldots, K \right\}$$

...to replace $I_K$ in the NUM problem.

$p = 4$

$p = 5$
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Bounding the Performance Gap

- There exists an optimal solution to the multipath problem sending positive rates on at most $N + L$ paths.
- For the linear utility case, the performance gap is bounded by

\[
\max \sum_{i=1}^{N} \rho \tilde{K}_i \\
\text{s. t.} \sum_{i=1}^{N} \tilde{K}_i \leq N + L, \\
0 \leq \tilde{K}_i \leq K_i, \tilde{K}_i \in \mathbb{Z}, \forall i = 1, \ldots, N.
\]
Optimal Rounding

- For a rate vector $x$, the optimal rounding $y$ of $x$ is a rate vector maximizing the total transmission rate such that the split ratio granularity constraints are satisfied and $y \leq x$.

- $\rho_K$ is the maximum throughput loss during the rounding for a user using $K$ paths.

- For the logarithmic utility case, $\rho_K$ is replaced by $\log \rho_K^R$, which is the maximum relative throughput loss during rounding.

$$\rho_K = \max_{\Gamma=p,\ldots,p+K-1} \frac{\Gamma - p}{\lceil \Gamma/K \rceil}, \quad \rho_K^R = \frac{K - 1}{p + K - 1}.$$
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Solving the Split Ratio Granularity Problem

- NP-hard to find the optimal solution to the split ratio granularity problem.
- The approach mentioned above provides an approximate solution based on the multipath problem.
- Can find a tighter relaxation that leads to a better performance guarantee.
Convex Relaxation

Replace the constraint set $S_K$ in the split ratio granularity problem by a convex set $T_K$ satisfying $S_K \subseteq T_K \subseteq I_K$:

$$T_K = \{ x \in I_K \| x \|_1 \leq C_K \}.$$  

Here $C_K$ is the maximum throughput over all rate vectors in $S_K$. 
Convex Relaxation

The performance guarantee for the approximation algorithm using convex relaxation is determined by $\rho^C_K$, which is the maximum throughput loss during the rounding for a rate vector in $\mathcal{T}_K$.

$\rho^C_K$ is either equal to $\rho_K$ or

$$\frac{K - 1}{p + K - 1}C_K.$$
Summary

Multipath Relaxation

Convex Relaxation

$\Delta_C$ $\Delta_M$

opt$_M$ − opt$_G$

Original Problem

Rounded Solution from Convex Relaxation

Rounded Solution from Multipath Relaxation
Thank You!