Network Utility Maximization with Path Cardinality Constraints

Yingjie Bi

School of Electrical and Computer Engineering
Cornell University

Joint work with Chee Wei Tan and Kevin Tang

Apr 12, 2016

Outline

Problem Formulation

Convex Relaxation

Throughput Maximization

Introduction

- The number of paths (W) allowed for each user in a routing protocol greatly affects:
  - Attainable performance;
  - Theoretical tractability;
  - Implementation complexity.
- Single-path routing ($W = 1$) and multipath routing ($W = \infty$) are two extreme cases.

Question

If W $(1 \leq W \leq \infty)$ paths are allowed for each user, what is the optimal routing performance and how to achieve it?

Model and Notation

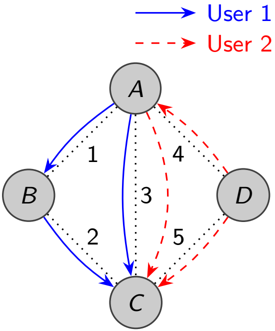
- Number of links $L = 5$. Number of users $N = 2$. Number of paths each user has $K^1 = K^2 = 2$.
- The paths of user i are represented by

$$R^1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad R^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

  Put $R^i_{lk} = 1$ if path k of user i passes through link l.

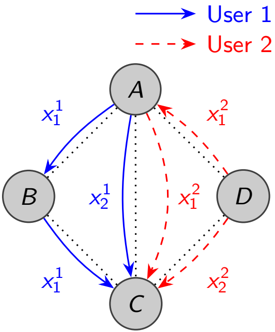
- The overall routing matrix

$$R = \begin{pmatrix} R^1 & R^2 \end{pmatrix}.$$



Model and Notation

- Link capacity $c = (c_1, \ldots, c_5)^T$.
- The kth entry x^i_k of vector x^i is the sending rate of user i on path k.
- Let
$$x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$
be the complete rate allocation.
- The utility $U^i(\cdot)$ of user i is a function of its total transmission rate $\|x^i\|_1$.



Problem Formulation

The *sparse routing* problem:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{N} U^i \left(\|x^i\|_1 \right) \\
\text{s.t.} \quad & Rx \leq c, \\
& x \geq 0, \\
& \|x^i\|_0 \leq W, \quad \forall i = 1, \ldots, N.
\end{aligned}
$$

- $\|x^i\|_0$: number of nonzero entries in x^i.
- opt_S: optimal value.
- Dropping the last N nonconvex constraints gives *multipath relaxation*.
- Stronger convex relaxation?

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An Observation

The sparse routing problem and the following have the same optimal value.

$$\begin{aligned} \max \quad & \sum_{i=1}^{N} U^i \left(\sum_{k=1}^{W} x^i_{[k]} \right) \\ \text{s. t.} \quad & Rx \leq c, \\ & x \geq 0. \end{aligned}$$

Here $(x^i_{[1]}, \ldots, x^i_{[K^i]})$ is a rearrangement of $(x^i_1, \ldots, x^i_{K^i})$ sorted in nonincreasing order, i.e., $x^i_{[1]} \geq \cdots \geq x^i_{[K^i]}$.

An Observation

$$\begin{aligned} \max \quad & \sum_{i=1}^{N} U^i \left(\sum_{k=1}^{W} x^i_{[k]} \right) \\ \text{s.t.} \quad & Rx \leq c, \\ & x \geq 0. \end{aligned}$$

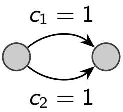
Define

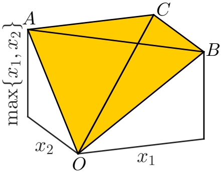
$$f^i(x^i) = \begin{cases} U^i \left(\sum_{k=1}^{W} x^i_{[k]} \right) & \text{if } 0 \leq x^i \leq \|c\|_\infty, \\ -\infty & \text{otherwise.} \end{cases}$$

The above problem can be rewritten as

$$\begin{aligned} \max \quad & \sum_{i=1}^{N} f^i(x^i) \\ \text{s.t.} \quad & Rx \leq c. \end{aligned}$$

Concave Envelope





One user transmits from the left to the right using a single path. Assume $U(s) = s$,

$$f(x) = \begin{cases} \max\{x_1, x_2\} & \text{if } 0 \le x \le 1, \\ -\infty & \text{otherwise.} \end{cases}$$

Consider the smallest concave function bounded below by f:

$$\hat{f}(x) = \begin{cases} x_1 + x_2 & \text{if } 0 \le x \le 1 \text{ and } x_1 + x_2 \le 1, \\ 1 & \text{if } 0 \le x \le 1 \text{ and } x_1 + x_2 > 1, \\ -\infty & \text{otherwise.} \end{cases}$$

Convex Relaxation

$$\max \quad \sum_{i=1}^{N} \hat{f}^i(x^i)$$
$$\text{s. t.} \quad Rx \leq c.$$

There exists an optimal solution $\hat{x}$ to the above convex relaxation satisfying

$$opt_S - \sum_{i=1}^{N} f^i(\hat{x}^i) \leq \sum_{i=1}^{\min\{N,L\}} \rho^i.$$

Here

$$\rho^i = \sup_{x^i} \left\{ \hat{f}^i(x^i) - f^i(x^i) \right\},$$

measures the non-concavity of the function f^i. We assume users are sorted in order $\rho^1 \geq \cdots \geq \rho^N$.

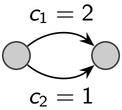
Outline

Throughput Maximization

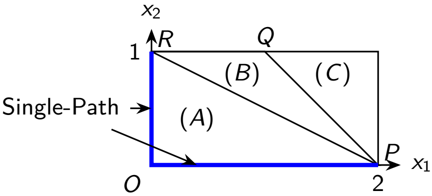
Restricting to the special case $U^i(s) = s$, the convex relaxation can be rewritten as

$$\begin{aligned} \max \quad & \sum_{i=1}^{N} \|x^i\|_1 \\ \text{s.t.} \quad & Rx \leq c, \\ & x \geq 0, \\ & \|x^i\|_1 \leq W\|c\|_\infty, \quad \forall i = 1, \ldots, N. \end{aligned}$$

Example



(A): Improved Convex Relaxation
$(A) + (B)$: Convex Relaxation
$(A) + (B) + (C)$: Multipath Relaxation



Line PQ: $x_1 + x_2 = 2$
Line PR: $x_1/2 + x_2 = 1$

Improved Convex Relaxation

$$\begin{aligned} \max \quad & \sum_{i=1}^{N} \|x^i\|_1 \\ \text{s. t.} \quad & Rx \leq c, \\ & x \geq 0, \\ & \sum_{k=1}^{K^i} \frac{x_k^i}{\hat{c}_k^i} \leq W, \quad \forall i = 1, \ldots, N. \end{aligned}$$

- $\hat{c}_k^i$: the minimum link capacity along the path k of user i.
- opt_C: optimal value.

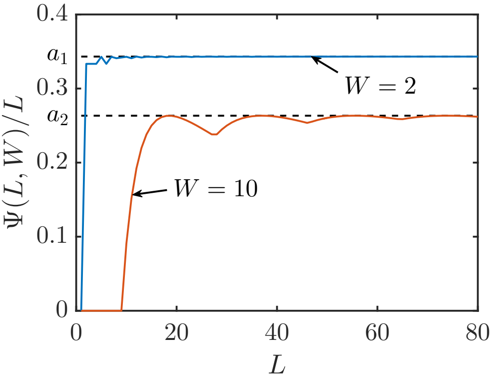
Performance Guarantee

Assume x is a vertex optimal solution to the above convex relaxation. Let y be the projection of x by picking up the W largest rates for each user. Then

$$opt_S - \sum_{i=1}^{N} \|y^i\|_1 \leq \Psi(L, W)\|c\|_\infty,$$

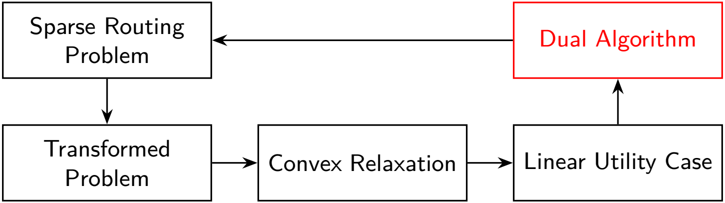
where $\Psi(L, W)$ is a constant depending on L and W.

Performance Guarantee



$$\Psi(L, W) = \max_{n=1,\ldots,\lfloor L/W \rfloor} \left(n - \frac{Wn^2}{n+L} \right) W.$$

Distributed Dual Algorithm



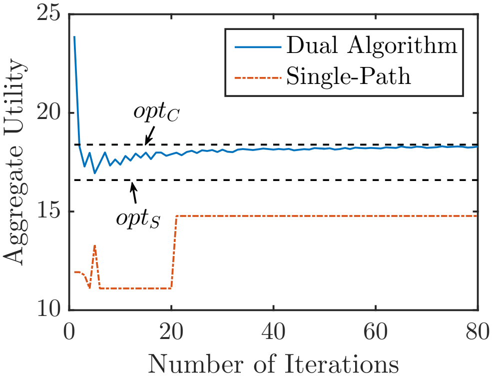
Distributed Dual Algorithm

The distributed dual algorithm converges to a vertex feasible solution $\hat{z}$ of the improved convex relaxation. Let $\hat{y}$ be the projection of $\hat{z}$ to a sparse routing configuration, then

$$opt_S - \sum_{i=1}^{N} \|\hat{y}^i\|_1 \leq \Psi(L, W)\|c\|_\infty + b\left(\frac{L}{W} + L\right)\|c\|_\infty.$$

Here b is a parameter in the algorithm.

Numerical Example



Further Directions

- Analyze the convergence rate of the distributed algorithm.
- Understand how the parameters in the distributed algorithm affects its performance.
- See whether there are stronger results for other utility functions.
- Generalize our result to the network cost minimization formulation.