Network Utility Maximization with Path Cardinality Constraints

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Outline

Problem Formulation

Convex Relaxation

Throughput Maximization
Introduction

- The number of paths ($W$) allowed for each user in a routing protocol greatly affects:
  - Attainable performance;
  - Theoretical tractability;
  - Implementation complexity.

- Single-path routing ($W = 1$) and multipath routing ($W = \infty$) are two extreme cases.
Question

If \( W (1 \leq W \leq \infty) \) paths are allowed for each user, what is the optimal routing performance and how to achieve it?
Model and Notation

- Number of links $L = 5$. Number of users $N = 2$. Number of paths each user has $K^1 = K^2 = 2$.
- The paths of user $i$ are represented by
  
  \[ R^1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad R^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

  Put $R^i_{lk} = 1$ if path $k$ of user $i$ passes through link $l$.
- The overall routing matrix
  
  \[ R = \begin{pmatrix} R^1 & R^2 \end{pmatrix}. \]
Model and Notation

- Link capacity $c = (c_1, \ldots, c_5)^T$.
- The $k$th entry $x^i_k$ of vector $x^i$ is the sending rate of user $i$ on path $k$.
- Let
  \[ x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \]
  be the complete rate allocation.
- The utility $U^i(\cdot)$ of user $i$ is a function of its total transmission rate $\|x^i\|_1$. 
Problem Formulation

The *sparse routing* problem:

\[
\begin{align*}
\text{max} \quad & \sum_{i=1}^{N} U^i \left( \|x^i\|_1 \right) \\
\text{s.t.} \quad & Rx \leq c, \\
& x \geq 0, \\
& \|x^i\|_0 \leq W, \quad \forall i = 1, \ldots, N.
\end{align*}
\]

\(\|x^i\|_0\): number of nonzero entries in \(x^i\).

\(\text{opt}_S\): optimal value.

Dropping the last \(N\) nonconvex constraints gives *multipath relaxation*.

Stronger convex relaxation?
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An Observation

The sparse routing problem and the following have the same optimal value.

\[
\max \sum_{i=1}^{N} U^i \left( \sum_{k=1}^{W} x^i_{[k]} \right)
\]

s.t. \(Rx \leq c,\)
\[x \geq 0.\]

Here \((x^i_{[1]}, \ldots, x^i_{[K^i]})\) is a rearrangement of \((x^i_1, \ldots, x^i_{K^i})\) sorted in nonincreasing order, i.e., \(x^i_{[1]} \geq \cdots \geq x^i_{[K^i]}\).
An Observation

\[
\begin{align*}
\max & \quad \sum_{i=1}^{N} U^i \left( \sum_{k=1}^{W} x^i_{[k]} \right) \\
\text{s.t.} & \quad Rx \leq c, \\
& \quad x \geq 0.
\end{align*}
\]

Define

\[
f^i(x^i) = \begin{cases} 
U^i \left( \sum_{k=1}^{W} x^i_{[k]} \right) & \text{if } 0 \leq x^i \leq \|c\|_\infty, \\
-\infty & \text{otherwise.}
\end{cases}
\]

The above problem can be rewritten as

\[
\max \sum_{i=1}^{N} f^i(x^i) \\
\text{s.t.} \quad Rx \leq c.
\]
Concave Envelope

One user transmits from the left to the right using a single path. Assume \( U(s) = s \),

\[
f(x) = \begin{cases} 
\max\{x_1, x_2\} & \text{if } 0 \leq x \leq 1, \\
-\infty & \text{otherwise.}
\end{cases}
\]

Consider the smallest concave function bounded below by \( f \):

\[
\hat{f}(x) = \begin{cases} 
x_1 + x_2 & \text{if } 0 \leq x \leq 1 \text{ and } x_1 + x_2 \leq 1, \\
1 & \text{if } 0 \leq x \leq 1 \text{ and } x_1 + x_2 > 1, \\
-\infty & \text{otherwise.}
\end{cases}
\]
Convex Relaxation

\[
\begin{align*}
\max & \quad \sum_{i=1}^{N} \hat{f}^i(x^i) \\
\text{s.t.} & \quad Rx \leq c.
\end{align*}
\]

There exists an optimal solution \( \hat{x} \) to the above convex relaxation satisfying

\[
\text{opts} - \sum_{i=1}^{N} f^i(\hat{x}^i) \leq \min\{N, L\} \sum_{i=1}^{N} \rho^i.
\]

Here

\[
\rho^i = \sup_{x^i} \left\{ \hat{f}^i(x^i) - f^i(x^i) \right\},
\]

measures the non-concavity of the function \( f^i \). We assume users are sorted in order \( \rho^1 \geq \cdots \geq \rho^N \).
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Convex Relaxation

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Throughput Maximization

Restricting to the special case $U^i(s) = s$, the convex relaxation can be rewritten as

$$
\max \sum_{i=1}^{N} \|x^i\|_1 \\
\text{s. t.} \quad Rx \leq c, \\
x \geq 0, \\
\|x^i\|_1 \leq W\|c\|_{\infty}, \quad \forall i = 1, \ldots, N.
$$
Example

\[ c_1 = 2 \]
\[ c_2 = 1 \]

- \((A)\): Improved Convex Relaxation
- \((A) + (B)\): Convex Relaxation
- \((A) + (B) + (C)\): Multipath Relaxation

**Single-Path**

- Line \(PQ\): \(x_1 + x_2 = 2\)
- Line \(PR\): \(x_1/2 + x_2 = 1\)
Improved Convex Relaxation

\[
\begin{align*}
\max & \quad \sum_{i=1}^{N} \| x^i \|_1 \\
\text{s.t.} & \quad Rx \leq c, \\
& \quad x \geq 0, \\
& \quad \sum_{k=1}^{K^i} \frac{x_k^i}{\hat{c}_k^i} \leq W, \quad \forall i = 1, \ldots, N.
\end{align*}
\]

- \( \hat{c}_k^i \): the minimum link capacity along the path \( k \) of user \( i \).
- \( \text{opt}_C \): optimal value.
Performance Guarantee

Assume $x$ is a vertex optimal solution to the above convex relaxation. Let $y$ be the projection of $x$ by picking up the $W$ largest rates for each user. Then

$$
\text{opt}_S - \sum_{i=1}^{N} \|y^i\|_1 \leq \Psi(L, W) \|c\|_\infty,
$$

where $\Psi(L, W)$ is a constant depending on $L$ and $W$. 
Performance Guarantee

\[ \Psi(L, W) = \max_{n=1, \ldots, \lfloor L/W \rfloor} \left( n - \frac{Wn^2}{n + L} \right) W. \]
Distributed Dual Algorithm

- Sparse Routing Problem
- Transformed Problem
- Convex Relaxation
- Linear Utility Case
- Dual Algorithm
The distributed dual algorithm converges to a vertex feasible solution $\hat{z}$ of the improved convex relaxation. Let $\hat{y}$ be the projection of $\hat{z}$ to a sparse routing configuration, then

$$\text{opt}_S - \sum_{i=1}^{N} \|\hat{y}^i\|_1 \leq \Psi(L, W)\|c\|_{\infty} + b \left(\frac{L}{W} + L\right) \|c\|_{\infty}.$$ 

Here $b$ is a parameter in the algorithm.
Numerical Example

![Graph showing aggregate utility over iterations for Dual Algorithm and Single-Path. The graph includes points labeled as $opt_C$ and $opt_S$.]
Further Directions

- Analyze the convergence rate of the distributed algorithm.
- Understand how the parameters in the distributed algorithm affects its performance.
- See whether there are stronger results for other utility functions.
- Generalize our result to the network cost minimization formulation.