Distributed Frequency-Preserving Optimal Load Control

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Abstract: Frequency control is traditionally done on the generation side. Recently we have formulated an optimal load control (OLC) problem and derived a load-side primary frequency control as a primal-dual solution to OLC. The load-side control rebalances power and resynchronize frequencies after a disturbance but cannot restore the nominal frequency. In this paper we modify OLC and derive from it a frequency-preserving load-side control that rebalances power and restores the nominal frequency after a disturbance. Unlike the generation-side secondary frequency control that is centralized, our load-side control only requires each bus to communicate a Lagrange multiplier with its neighbors. We prove that such a distributed load-side control is globally asymptotically stable and illustrate its convergence with simulation.

Keywords: Smart grids, Load frequency control, Optimal operation and control of power systems, Power systems stability.

1. INTRODUCTION

Frequency control maintains the frequency of a power network at its nominal value when demand or supply fluctuates. It is traditionally implemented on the generation side and consists of three mechanisms that work in concert (Wood and Wollenberg, 1996; Bergen and Vittal, 2000; Machowski et al., 2008). The primary frequency control, called the droop control and completely decentralized, operates on a timescale up to low tens of seconds and uses a governor to adjust, around a setpoint, the mechanical power input to a generator based on the local frequency deviation. The primary control can rebalance power and stabilize the frequency but does not restore the nominal frequency. The secondary frequency control (called automatic generation control) operates on a timescale up to a minute or so and adjusts the setpoints of governors in a control area in a centralized fashion to drive the frequency back to its nominal value and the inter-area power flows to their scheduled values. Economic dispatch operates on a timescale of several minutes or up and schedules the output levels of generators that are online and the inter-area power flows. See (Ilic, 2007; Kiani and Annaswamy, 2012) for a recent hierarchical model of power systems and their stability analysis.

Load-side participation in frequency control offers many advantages, including faster response, lower fuel consumption and emission, and better localization of disturbances. The idea of using frequency adaptive loads dates back to (Schweppe et al., 1980) that advocates their large scale deployment to “assist or even replace turbine-governed systems and spinning reserve.” They also proposed to use spot prices to incentivize the users to adapt their consumption to the true cost of generation at the time of consumption. Remarkably it was emphasized back then that such frequency adaptive loads will “allow the system to accept more readily a stochastically fluctuating energy source, such as wind or solar generation” (Schweppe et al., 1980). This is echoed recently in, e.g., (Trudnowski et al., 2006; Lu and Hammerstrom, 2006; Short et al., 2007; Donnelly et al., 2010; Brooks et al., 2010; Callaway and Hiskens, 2011; Molina-Garcia et al., 2011) that argue for “grid-friendly” appliances, such as refrigerators, water or space heaters, ventilation systems, and air conditioners, as well as plug-in electric vehicles to help manage energy imbalance. Simulations in all these studies have consistently shown significant improvement in performance and reduction in the need for spinning reserves. A small scale project by the Pacific Northwest National Lab in 2006–2007 demonstrated the use of 200 residential appliances in primary frequency control that automatically reduce their consumption when the frequency of the household dropped below a threshold (59.95Hz) (Hammerstrom et al., 2007).

Despite these simulation studies and field trials, there is not much analytic study of how large-scale deployment of distributed load-side frequency control will behave, with (Kiani and Annaswamy, 2012) as a notable exception. Recently another model is presented in (Zhao et al., 2013) that formulates an optimal load control (OLC) problem where the objective is to minimize the aggregate disutility of tracking an operating point subject to power balance over the network. The main conclusion is that frequency-based load control, coupled with the power network dynamics, serves as a primal-dual algorithm to solve (the Lagrangian dual of) OLC. It establishes the stability of completely decentralized load participation in primary frequency control. Like the droop control on the generation side, the scheme in (Zhao et al., 2013) rebalances power and resynchronizes frequencies after a disturbance, but does not drive the new system frequency to its nominal value. The goal of this paper is to extend the scheme of (Zhao et al., 2013) to restore the nominal frequency after a disturbance.

In (Zhao et al., 2013) an optimization problem (OLC) is first designed and the load control is then derived as a primal-dual algorithm to solve its Lagrangian dual problem. We follow the same model here and modify OLC such that its optimal solution restores the nominal frequency (Section 3). We then derive our load control scheme as the primal-dual algorithm for the modified OLC (Section 4). Unlike the control in (Zhao et al., 2013) that is completely decentralized, our scheme requires each bus to communicate a Lagrange multiplier with its neighbors. This is not surprising considering AGC that coordinates the generators within each control area is centralized. We prove that our design is globally asymptotically stable and converges to an optimal solution of the modified OLC (Section 5). Finally we present preliminary simulations to illustrate these results (Section 6).

2. PRELIMINARIES

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{N}$ the set of natural numbers. Given a finite set $S \subset \mathbb{N}$ we use $|S|$ to denote its cardinality. For a set of scalar numbers $a_i \in \mathbb{R}$, $i \in S$ we denote a_S as the column vector of the a_i components, i.e. $a_S := (a_i, \in S) \in \mathbb{R}^{|S|}$; we usually drop the subscript S when set is known from the context. Similarly, for two vectors $a \in \mathbb{R}^{|S|}$ and $b \in \mathbb{R}^{|S'|}$ we define the column vector $x = (a, b) \in \mathbb{R}^{|S|+|S'|}$. Given any matrix A, we denote its transpose as A^T. The diagonal matrix of a vector a is represented by $\text{diag}(a_i)$ and for a set of matrices $\{A_i,\ i \in S\}$ we let $\text{blockdiag}(A_i)$ denote the block diagonal matrix. Finally, we use $\mathbf{1}$ ($\mathbf{0}$) to denote the vector of all ones (zeros).

2.1 Network Model

We consider a power network described by the graph $G(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, ..., |\mathcal{N}|\}$ is the set of buses and $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is the set of transmission lines denoted by either e or ij such that if $ij \in \mathcal{E}$, then $ji \notin \mathcal{E}$. We partition the buses $\mathcal{N} = \mathcal{G} \cup \mathcal{L}$ and use $\mathcal{G}$ and $\mathcal{L}$ to denote the set of generator and load buses respectively.

The evolution of the transmission network is described by

$$M_i \dot{\omega}_i = P_i^m - (d_i + \hat{d}_i) - \sum_{e \in \mathcal{E}} C_{ie} P_e \qquad i \in \mathcal{G} \tag{1a}$$

$$0 = P_i^m - (d_i + \hat{d}_i) - \sum_{e \in \mathcal{E}} C_{ie} P_e \qquad i \in \mathcal{L} \tag{1b}$$

$$\dot{P}_{ij} = B_{ij}(\omega_i - \omega_j) \qquad ij \in \mathcal{E} \tag{1c}$$

where d_i denotes an aggregate controllable load, $\hat{d}_i := D_i \omega_i$ denotes an aggregate uncontrollable but frequency-sensitive load as well as damping loss at generator i, M_i is the generator's inertia, P_i^m is the mechanical power injected by a generator $i \in \mathcal{G}$, $-P_i^m$ is the aggregate power consumed by constant loads for $i \in \mathcal{L}$, and P_{ij} is the line real power flow from i to j. Finally, C_{ie} are the elements of the incidence matrix $C \in \mathbb{R}^{|\mathcal{N}| \times |\mathcal{E}|}$ of the graph G such that $C_{ie} = -1$ if $e = ji \in \mathcal{E}$, $C_{ie} = 1$ if $e = ij \in \mathcal{E}$ and $C_{ie} = 0$ otherwise. We refer the reader to (Zhao et al., 2013) for a detailed motivation of the model.

Suppose the system (1) is in equilibrium, i.e., $\dot{\omega}_i = 0$ for all i and $\dot{P}_{ij} = 0$ for all ij. We further assume that the equilibrium frequencies ω_i^* are synchronized at the nominal value and the equilibrium branch power flows P_{ij}^* are equal to their scheduled values. We consider the system locally around this equilibrium point (ω^*, P^*) and henceforth all variables in (1) denote perturbations around (ω^*, P^*). Suppose at time 0, there is a disturbance represented by the vector (perturbations) $P^m := (P_i^m, i \in \mathcal{G} \cup \mathcal{L})$ that produces a power imbalance.

Then, we are interested in designing a distributed control mechanism that rebalances the system while preserving the frequency in its nominal value. Furthermore, we would like this mechanism to be fair among all the users (or loads) that are willing adapt.

2.2 Optimal Load Control

In (Zhao et al., 2013) it was shown that given load cost function $c_i(d_i)$ for the frequency insensitive loads and $\frac{\hat{d}_i^2}{2D_i}$ for the frequency sensitive loads, then by setting

$$d_i(\omega_i) = c_i^{'-1}(\omega_i) \tag{2}$$

makes (1) converge to the optimal value of following optimal load control (OLC) problem

OLC:

$$\underset{d, \hat{d}}{\text{minimize}} \qquad \sum_{i \in \mathcal{N}} c_i(d_i) + \frac{\hat{d}_i^2}{2D_i} \tag{3a}$$

subject to

$$\sum_{i \in \mathcal{N}} (d_i + \hat{d}_i) = \sum_{i \in \mathcal{N}} P_i^m. \tag{3b}$$

The key insight of (Zhao et al., 2013) is that one can define a distributed dual problem of OLC such that the Lagrange multipliers ν_i satisfy $\hat{d}_i^* = D_i \nu_i^*$ for the optimal solution. Therefore, since $\hat{d}_i := D_i \omega_i$, once can think of the frequency deviation ω_i, for $i \in \mathcal{N}$, as the Lagrange multiplier ν_i of a (slightly modified) version of OLC.

However, the system (1)-(2) suffers from the disadvantage that once a perturbation is included on P^m, the resulting equilibrium point has $\omega_i^* = \omega^* \neq 0$. This is because the optimal solution of OLC is achieved with $\omega^* \neq 0$. Thus, in order to achieve distributed load control without shifting the frequency one needs to modify the OLC in order to impose the constraint $\omega_i = 0$.

3. FREQUENCY PRESERVING OLC

We now proceed to describe our frequency-preserving optimal load control problem. The crux of our solution comes from designing an optimal load control problem in which the optimal solutions satisfies

$$\hat{d}_i^* = D_i \nu_i^*, \quad \hat{d}_i^* = 0, \tag{4}$$

while still preserving the flow constraints (3b). That is, we want to formulate an Optimal Load Control problem in which the frequency deviations still represent Lagrange multipliers, yet the optimal solution has $\omega_i^* = \hat{d}_i^* / D_i = 0$. Therefore, any distributed control derived from this optimization problem will restore the frequency to its nominal value after a disturbance.

We therefore propose the following Frequency-Preserving Optimal Load Control (FP-OLC) problem

FP-OLC:

$$\underset{d,\hat{d},P,R}{\text{minimize}} \quad \sum_{i\in\mathcal{N}} c_i(d_i) + \frac{\hat{d}_i^2}{2D_i} \tag{5a}$$

subject to

$$P_i^m - (d_i + \hat{d}_i) = \sum_{e\in\mathcal{E}} C_{ie}P_e, \quad i\in\mathcal{N} \tag{5b}$$

$$P_i^m - d_i = \sum_{e\in\mathcal{E}} C_{ie}R_e, \quad i\in\mathcal{N} \tag{5c}$$

where (5c) is an additional constraint that will be shown to guarantee the second equality of (4).

Assumption 1. (Strict Convexity). The cost function $c_i(d_i)$ is strictly convex and second order differentiable ($c_i \in C^2$) in the interior of its domain $\mathcal{D}_i := [\underline{d}_i, \overline{d}_i] \subseteq \mathbb{R}$, such that $c_i(d_i) \to +\infty$ whenever $d_i \to \partial\mathcal{D}_i$.

Assumption 2. (Slater Condition). The PF-OLC problem (5) has a strictly feasible point such that $\underline{d}_i < d_i < \overline{d}_i$.

In order to make presentation compact sometimes we will use $x = (P, R) \in \mathbb{R}^{2|\mathcal{L}|}$ and $\sigma = (\nu, \lambda) \in \mathbb{R}^{2|\mathcal{N}|}$, as well as $\sigma_i = (\nu_i, \lambda_i)$. To understand how (4) is satisfied in FP-OLC, we consider the dual function $D(\sigma)$ of the FP-OLC problem.

$$D(\sigma) = \inf_{d,\hat{d},x} L(d, \hat{d}, x, \sigma) \tag{6}$$

where

$$\begin{aligned} L(d,\hat{d},x,\sigma) &= \sum_{i\in\mathcal{N}} \Bigg(c_i(d_i) + \frac{\hat{d}_i^2}{2D_i} + \nu_i(P_i^m - (d_i + \hat{d}_i) \\ &\quad - \sum_{e\in\mathcal{E}} C_{ie}P_e) + \lambda_i(P_i^m - d_i - \sum_{e\in\mathcal{E}} C_{ie}R_e) \Bigg) \\ &= \sum_{i\in\mathcal{N}} \Bigg(c_i(d_i) - \nu_i d_i + \frac{\hat{d}_i^2}{2D_i} - \nu_i\hat{d}_i + (\nu_i + \lambda_i)P_i^m \Bigg) \\ &\quad + \sum_{ij\in\mathcal{E}} ((\nu_j - \nu_i)P_{ij} + (\lambda_j - \lambda_i)R_{ij}) \end{aligned} \tag{7}$$

Since $c_i(d_i)$ and $\frac{\hat{d}_i^2}{2D_i}$ are radially unbounded, the minimization over d and $\hat{d}$ in (6) is always finite for given x and σ. However, whenever $\nu_i \neq \nu_j$ or $\lambda_i \neq \lambda_j$ for $ij \in \mathcal{E}$, then one can modify P_{ij} or R_{ij} to arbitrarily decrease (6). Thus, the infimum is attained if and only if

$$\nu_i = \nu_j \quad \text{and} \quad \lambda_i = \lambda_j, \quad \forall i \in \mathcal{N}. \tag{8}$$

Moreover, the minimum value must satisfy

$$c_i'(d_i) = \nu_i + \lambda_i \quad \text{and} \quad \frac{\hat{d}_i}{D_i} = \nu_i, \quad \forall i \in \mathcal{N}. \tag{9}$$

Using (8) and (9) we can compute the dual function

$$D(\sigma) = \begin{cases} \sum_{i\in\mathcal{N}} \Phi_i(\sigma_i) & \sigma \in \tilde{N} \\ -\infty & \text{otherwise,} \end{cases} \tag{10}$$

where

$$\Phi_i(\sigma_i) = c_i(d_i(\sigma_i)) + (\nu_i + \lambda_i)(P_i^m - d_i(\sigma_i)) - \frac{D_i}{2}\nu_i^2, \tag{11}$$

with

$$d_i(\sigma_i) = c_i'^{-1}(\nu_i + \lambda_i) \tag{12}$$

and

$$\tilde{N} := \{(\sigma) \in \mathbb{R}^{2|\mathcal{N}|} : \nu_j = \nu_i,\ \lambda_j = \lambda_i\ \forall ij \in \mathcal{E}\}.$$

One interesting property of (10) is that although $D(\sigma)$ does not seem to be decoupled among the buses due to the terms involving P_{ij} and R_{ij}, the structure of C_{ie} makes them vanish and decouples the minimization in (6).

The dual problem of the FP-OLC (DFP-OLC) is then given by

DFP-OLC:

$$\underset{\nu,\lambda}{\text{maximize}} \quad D(\sigma) = \sum_{i\in\mathcal{N}} \Phi_i(\nu_i, \lambda_i) \tag{13a}$$

subject to

$$\nu_j - \nu_i = 0, \quad ij \in \mathcal{E} \tag{13b}$$

$$\lambda_j - \lambda_i = 0, \quad ij \in \mathcal{E} \tag{13c}$$

Remark 1. When the multipliers λ_i and equation (13c) are removed, DFP-OLC is exactly the distributed optimal load control proposed in (Zhao et al., 2013). This implies that the solution (1)-(2) can be interpreted as a partial primal-dual algorithm that solves (5) with the constraint (5c) removed. [1]

Although $D(\sigma)$ is only finite on $\tilde{N}$, $\Phi_i(\sigma_i)$ is finite everywhere on $\mathbb{R}^2$. Thus sometimes we use the extended version of the dual function

$$\Phi(\sigma) = \sum_{i\in\mathcal{N}} \Phi_i(\sigma_i) \tag{14}$$

instead of $D(\sigma)$, knowing that $D(\sigma) = \Phi(\sigma)$ for $\sigma \in \tilde{N}$.

The following two lemmas describe several properties of our optimization problem. Given any set $S \subset \mathcal{N}$ we define

$$\Phi_S(\sigma) := \sum_{i\in S} \Phi_i(\sigma_i).$$

Lemma 2. (Strict concavity of $\Phi_S(\sigma)$). For any nonempty set S, the function $\Phi_S(\sigma)$ is the sum of strictly concave functions $\Phi_i(\sigma_i)$ and it is therefore strictly concave. In particular, the (extended) dual function $\Phi(\sigma) = \Phi_{\mathcal{N}}(\sigma)$ is strictly concave.

Proof. From the derivation of $\Phi_i(\sigma_i)$ it is easy to show that

$$\Phi_i(\sigma_i) = \min_{d_i,\hat{d}_i} L_i(d_i, \hat{d}_i, \sigma_i) \tag{15}$$

where

$$L_i(d_i, \hat{d}_i, \sigma_i) := c_i(d_i) + \frac{\hat{d}_i^2}{2D_i} + (\nu_i + \lambda_i)(P_i^m - d_i) - \nu_i\hat{d}_i.$$

Notice that $L_i(d_i, \hat{d}_i, \sigma_i)$ is linear in σ_i and strictly convex in $(d_i, \hat{d}_i)$.

Let $d_i(\sigma_i)$ and $\hat{d}_i(\sigma_i)$ be the unique minimizer of (15). Then from (9) it follows that $d_i(\sigma_i) = d_i(\nu_i + \lambda_i) = c_i'^{-1}(\nu_i + \lambda_i)$ and $\hat{d}_i(\sigma_i) = D_i\nu_i$.

We will first show that, given $\sigma_{i,1} \neq \sigma_{i,2}$, then

$$(d_i(\sigma_{i,1}), \hat{d}_i(\sigma_{i,1})) \neq (d_i(\sigma_{i,2}), \hat{d}_i(\sigma_{i,2})). \tag{16}$$

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[1] We the algorithm a *partial* whenever some of the either primal or dual variables are statically set to its Lagrangian optimal value, e.g. (2).

Suppose by contradiction that there is $\sigma_{i,1} \neq \sigma_{i,2}$ such that

$$(d_i(\sigma_{i,1}), \hat{d}_i(\sigma_{i,1})) = (d_i(\sigma_{i,2}), \hat{d}_i(\sigma_{i,2})).$$

Then by (9), $c_i'^{-1}(\nu_{i,1} + \lambda_{i,1}) = c_i'^{-1}(\nu_{i,2} + \lambda_{i,2})$ and $D_i\nu_{i,1} = D_i\nu_{i,2}$. Thus, $\nu_{i,1} = \nu_{i,2} = \nu$ and $c_i'^{-1}(\nu + \lambda_{i,1}) = c_i'^{-1}(\nu + \lambda_{i,2})$. But since $c_i(\cdot)$ is strictly convex, c_i' and its inverse are strictly increasing which implies that $\lambda_{i,1} = \lambda_{i,2} = \lambda$. Contradiction.

Finally, let $\theta \in [0, 1]$ and consider any two $\sigma_{i,1} \neq \sigma_{i,2}$. Then,

$$\begin{aligned}
&\Phi_i(\theta\sigma_{i,1} + (1-\theta)\sigma_{i,2}) = \min_{d_i,\hat{d}_i} L_i(d_i, \hat{d}_i, \theta\sigma_{i,1} + (1-\theta)\sigma_{i,2})\\
&= \min_{d_i,\hat{d}_i}\ \theta L_i(d_i, \hat{d}_i, \sigma_{i,1}) + (1-\theta)L_i(d_i, \hat{d}_i, \sigma_{i,2})\\
&> \theta \min_{d_i,\hat{d}_i}\ L_i(d_i, \hat{d}_i, \sigma_{i,1}) + (1-\theta) \min_{d_i,\hat{d}_i}\ L_i(d_i, \hat{d}_i, \sigma_{i,2})\\
&= \theta\ \Phi_i(\sigma_{i,1}) + (1-\theta)\Phi_i(\sigma_{i,2})
\end{aligned}$$

where the strict inequality follows from (16). Thus, $\Phi_i(\sigma_i)$ is strictly concave and by using the definition of strict concavity we get $\Phi_S(\sigma)$ is strictly concave $\forall S \subseteq \mathcal{N}$. □

Lemma 3. (FP-OLC Optimality). Given a connected graph $(\mathcal{N}, \mathcal{E})$, then there exists scalars ν^* and λ^* such that $(d^*, \hat{d}^*, x^*, \sigma^*)$ is a primal-dual optimal solution of FP-OLC and DFP-OLC if and only if $(d^*, \hat{d}^*, x^*)$ is primal feasible,

$$\hat{d}_i^* = D_i\nu_i^*, \quad d_i^* = c_i'^{-1}(\nu_i^* + \lambda_i^*), \tag{17}$$

$$\nu_i^* = \nu^* \ \text{ and } \ \lambda_i^* = \lambda^*. \tag{18}$$

Moreover, d^*, $\hat{d}^*$ and $\sigma^* = (\nu^*, \lambda^*)$ are unique and $\nu^* = 0$.

Proof. Assumptions 1 and 2 guarantee that the solution to the primal (PF-OLC) is finite. Moreover, since by Assumption 2 there is a feasible $d \in \text{int}\mathcal{D} = \Pi_{i=1}^{|\mathcal{N}|}\mathcal{D}_i$, then the Slater condition is satisfied (Boyd and Vandenberghe, 2004) and there is zero duality gap.

Thus, since FP-OLC only has linear equality constraints, we can use Karush-Kuhn-Tucker (KKT) conditions (Boyd and Vandenberghe, 2004) to characterize the primal dual optimal solution. Thus $(d^*, \hat{d}^*, x^*, \sigma^*)$ is primal dual optimal if and only if:

(i) Primal feasibility: (5b)-(5c)
(ii) Dual feasibility: (13b)-(13c)
(iii) Stationarity:

$$\frac{\partial}{\partial d}L(d, \hat{d}, x, \sigma) = 0, \frac{\partial}{\partial \hat{d}}L(d, \hat{d}, x, \sigma) = 0$$
$$\text{and } \frac{\partial}{\partial x}L(d, \hat{d}, x, \sigma) = 0$$

Primal feasibility (i) is satisfied by assumption. Dual feasibility (ii) amounts to $\nu_i^* = \nu_j^*$ and $\lambda_i^* = \lambda_j^*$ $\forall ij \in \mathcal{E}$, which since G is connected is obtained if and only ifs

$$\nu_i^* = \nu^* \text{ and } \lambda_i^* = \lambda^* \quad \forall i \in \mathcal{N}.$$

Finally, using (7), Stationarity (iii) is equivalent to (ii) and

$$\frac{\partial L}{\partial d_i}(d^*, \hat{d}^*, x^*, \sigma^*) = c_i'(d_i^*) - (\nu^* + \lambda^*) = 0 \tag{19a}$$

$$\frac{\partial L}{\partial \hat{d}_i}L(d^*, \hat{d}^*, x^*, \sigma^*) = \frac{\hat{d}_i^*}{D_i} - \nu^* = 0 \tag{19b}$$

which are the same as (17).

To show $\nu^* = 0$ we use (i). Adding (5b) over $i \in \mathcal{N}$ gives

$$\begin{aligned}
0 &= \sum_{i\in\mathcal{N}}\left(P_i^m - (d_i^* + \hat{d}_i^*) - \sum_{e\in\mathcal{E}} C_{ie}P_e\right)\\
&= \sum_{i\in\mathcal{N}}\left(P_i^m - (d_i^* + \hat{d}_i^*)\right) - \sum_{e=ij\in\mathcal{E}}(C_{ie}P_e + C_{je}P_e)\\
&= \sum_{i\in\mathcal{N}}\left(P_i^m - (d_i^* + \hat{d}_i^*)\right)
\end{aligned}\tag{20}$$

and similarly (5c) gives

$$0 = \sum_{i\in\mathcal{N}} P_i^m - d_i^* \tag{21}$$

Thus, subtracting (20) from (21) gives

$$0 = \sum_{i\in\mathcal{N}}\hat{d}_i^* = \sum_{i\in\mathcal{N}} D_i\nu^* = \nu^*\sum_{i\in\mathcal{N}} D_i$$

and since $D_i > 0$ $\forall i \in \mathcal{N}$, it follows that $\nu^* = 0$. Finally, since by Lemma 2, $\Phi(\sigma)$ is strictly concave, then σ^* is unique and using (19a)-(19b) it follows that d^* and $\hat{d}^*$ are also unique since $c_i'(\cdot)$ is strictly increasing. □

4. DISTRIBUTED OPTIMAL LOAD CONTROL

In this section we show how we can leverage the power network dynamics to solve the FP-OLC problem in a distributed way. Our solution is based on the classical primal dual optimization algorithm that has been of great use to design congestion control mechanisms in communication networks (Kelly et al., 1998; Low and Lapsley, 1999; Srikant, 2004; Palomar and Chiang, 2006).

Let

$$\begin{aligned}
L(x, \sigma) &= \underset{d,\hat{d}}{\text{minimize}} \quad L(d, \hat{d}, x, \sigma)\\
&= L(d(\sigma), \hat{d}(\sigma), x, \sigma)\\
&= \Phi(\sigma) - \nu^T CP - \lambda^T CR
\end{aligned}\tag{22}$$

where $L(d, \hat{d}, x, \sigma)$ is defined as in (7), $d(\sigma) := (d_i(\sigma_i))$ and $\hat{d}(\sigma) := (\hat{d}_i(\sigma_i))$ according to (17).

Similarly to (Zhao et al., 2013) we propose the following partial primal-dual

$$\dot{\nu}_i = \zeta_i(P_i^m - (d_i(\sigma_i) + D_i\nu_i) - \sum_{e\in\mathcal{E}} C_{ie}P_e), i \in \mathcal{G} \tag{23a}$$

$$0 = P_i^m - (d_i(\sigma_i) + D_i\nu_i) - \sum_{e\in\mathcal{E}} C_{ie}P_e, \qquad i \in \mathcal{L} \tag{23b}$$

$$\dot{\lambda}_i = \gamma_i(P_i^m - d_i(\sigma_i) - \sum_{e\in\mathcal{E}} C_{ie}R_e), \qquad i \in \mathcal{N} \tag{23c}$$

$$\dot{P}_{ij} = \beta_{ij}(\nu_i - \nu_j), \qquad ij \in \mathcal{L} \tag{23d}$$

$$\dot{R}_{ij} = \alpha_{ij}(\lambda_i - \lambda_j), \qquad ij \in \mathcal{L} \tag{23e}$$

where its name comes from the fact that

$$\frac{\partial}{\partial \nu_i} L(x,\sigma) = P_i^m - (d_i(\sigma_i) + D_i\nu_i) - \sum_{e\in\mathcal{E}} C_{ie}P_e \quad (24a)$$

$$\frac{\partial}{\partial \lambda_i} L(x,\sigma) = P_i^m - d_i(\sigma_i) - \sum_{e\in\mathcal{E}} C_{ie}R_e \quad (24b)$$

$$\frac{\partial}{\partial P_{ij}} L(x,\sigma) = -(\nu_i - \nu_j) \quad (24c)$$

$$\frac{\partial}{\partial R_{ij}} L(x,\sigma) = -(\lambda_i - \lambda_j) \quad (24d)$$

Equations (23a), (23b) and (23d) show that dynamics (1) can be interpreted as a subset of the primal-dual dynamics described in (23) for the special case when $\gamma_i = M_i^{-1}$ and $\beta_{ij} = B_{ij}$. Therefore, we can interpret the frequency ω_i as the Lagrange multiplier ν_i.

This observation motivates us to propose a distributed load control scheme that is naturally decomposed between
Power Network Dynamics:

$$\dot{\omega}_i = M_i^{-1}(P_i^m - (d_i + \hat{d}_i) - \sum_{e\in\mathcal{E}} C_{ie}P_e) \quad i\in\mathcal{G} \quad (25a)$$

$$0 = P_i^m - (d_i + \hat{d}_i) - \sum_{e\in\mathcal{E}} C_{ie}P_e \quad i\in\mathcal{L} \quad (25b)$$

$$\dot{P}_{ij} = B_{ij}(\omega_i - \omega_j) \quad i\sim j \quad (25c)$$

$$\hat{d}_i = D_i\omega_i \quad i\in\mathcal{N} \quad (25d)$$

and
Dynamic Load Control:

$$\dot{\lambda}_i = \gamma_i(P_i^m - d_i - \sum_{e\in\mathcal{E}} C_{ie}R_e) \quad i\in\mathcal{N} \quad (26a)$$

$$\dot{R}_{ij} = \alpha_{ij}(\lambda_i - \lambda_j) \quad ij\in\mathcal{E} \quad (26b)$$

$$d_i = c_i'^{-1}(\omega_i + \lambda_i) \quad i\in\mathcal{N} \quad (26c)$$

Equations (25) and (26) show how the network dynamics can be complemented with dynamic load control such that the whole system amount to a distributed primal-dual algorithm that tries to find a saddle point on $L(x,\sigma)$. The next section shows that this system does achieve optimality as intended.

5. OPTIMALITY AND CONVERGENCE ANALYSIS

In this section we will show that the system (25)-(26) converges globally to an optimal solution of the FP-OLC problem (5). We will achieve this objective in two steps. First, we will show that every equilibrium point of (25)-(26) is an optimal solution of (5), and second, we will show that every trajectory $(d(t), \hat{d}_i(t), P(t), R(t), \omega(t), \lambda(t))$ converges to an equilibrium point of (25)-(26), or equivalently (23).

Theorem 4. (Optimality). Given a point $p^* = (d^*, \hat{d}^*, x^*, \sigma^*)$, then p^* is an equilibrium point of (25)-(26) if and only if is a primal dual optimal solution to the FP-OLC problem.

Proof. The proof of this theorem is a direct application of Lemma 3.

Let $(d^*, \hat{d}^*, x^*, \sigma^*)$ be an equilibrium point of (25)-(26). Then, it follows from (25c) and (26b) that

$$\omega_i^* = \omega^* \text{ and } \lambda_i^* = \lambda^*$$

for some scalars ω^* and λ^*, which implies that σ^* is a dual feasible point.

Moreover, since $\dot{\omega}_i = 0$ and $\dot{\lambda}_i = 0$, then (25a)-(25b) and (26a) imply that $(d^*, \hat{d}^*, P^*, R^*)$ is a primal feasible point. Finally, by definition of (25)-(26) condition (17) is always satisfied. Thus we are under the conditions of Lemma (3) and therefore $p^* = (d^*, \hat{d}^*, x^*, \sigma^*)$ is primal-dual optimal which also implies by (17) that $\omega^* = 0$.

Conversely, by the proof of Lemma 2 every optimal solution $p^* = (d^*, \hat{d}^*, x^*, \sigma^*)$ must satisfy the Karush Kuhn Tucker conditions which implies by (23) that p^* is an equilibrium point. □

Theorem 4 implies that every equilibrium solution of (25)-(26) is optimal with respect to FP-OLC. The rest of this section is devoted to showing that in fact for every initial condition $(P(0), R(0), \omega(0), \lambda(0))$, the system (25)-(26) converges to one of such optimal solution.

Since we showed in Section 4 that (25)-(26) are just a special case of (23), we will provide our convergence result for (23). Our global convergence proof leverages the results of (Feijer and Paganini, 2010) on global convergence in network flow control. Unfortunately, the results presented there cannot be readily applied as (23) is not a full primal-dual gradient law due to constraint (23b). However, the next lemma shows that (23) amounts to a primal-dual gradient law with respect to a different Lagrangian.

Lemma 5. (Primal-dual Gradient Law). Let $\sigma_{-\nu_\mathcal{L}} = (\nu_\mathcal{G}, \lambda)$ and consider the reduced Lagrangian

$$L(x, \sigma_{-\nu_\mathcal{L}}) = \underset{\nu_\mathcal{L}}{\text{maximize}}\ L(x,\sigma). \quad (27)$$

Then under Assumption 1, $L(x, \sigma_{-\nu_\mathcal{L}})$ is strictly concave in $\sigma_{-\nu_\mathcal{L}}$, convex in x and the dynamics (23) amount to

$$\dot{\sigma}_{-\nu_\mathcal{L}} = Z\frac{\partial}{\partial \sigma_{-\nu_\mathcal{L}}} L(x,\sigma_{-\nu_\mathcal{L}}) \text{ and } \dot{x} = -X\frac{\partial}{\partial x}L(x,\sigma_{-\nu_\mathcal{L}}) \quad (28)$$

where $Z = \text{blockdiag}(\text{diag}(\zeta_i)_{i\in\mathcal{G}}, \text{diag}(\gamma_i)_{i\in\mathcal{N}})$ and $X = \text{blockdiag}(\text{diag}(\beta_{ij})_{ij\in\mathcal{E}}, \text{diag}(\alpha_{ij})_{ij\in\mathcal{E}})$.

Proof. By Lemma 2 and (22), $L(x,\sigma)$ is strictly concave in σ. Therefore, it follows that there exists a unique minimizer $\nu_\mathcal{L}^*(x,\sigma_{-\nu_\mathcal{L}})$ of (27). Moreover, $\nu_\mathcal{L}^*(x,\sigma_{-\nu_\mathcal{L}})$ must satisfy

$$\frac{\partial L}{\partial \nu_\mathcal{L}}(x, \sigma_{-\nu_\mathcal{L}}, \nu_\mathcal{L}^*(x,\sigma_{-\nu_\mathcal{L}})) = \quad (29a)$$

$$= \frac{\partial \Phi_\mathcal{L}}{\partial \nu_\mathcal{L}}(\sigma_{-\nu_\mathcal{L}}, \nu_\mathcal{L}^*(x,\sigma_{-\nu_\mathcal{L}})) - (C_\mathcal{L}P)^T = 0 \quad (29b)$$

which is equivalent to (23b).

Therefore, we can apply the Envelope Theorem (Mas-Collel et al., 1995) on (27) to compute the partial derivatives of $L(x,\sigma_{-\nu_\mathcal{L}})$ with respect to $\nu_\mathcal{G}$, λ, P and R. Thus if we partition the incidence matrix between generator and load buses

$$C = \begin{bmatrix} C_\mathcal{G} \\ C_\mathcal{L} \end{bmatrix} \quad \text{and let} \quad \Phi(\sigma) = \Phi_\mathcal{G}(\sigma_\mathcal{G}) + \Phi_\mathcal{L}(\sigma_\mathcal{L})$$

where $\Phi_\mathcal{G}(\sigma_\mathcal{G}) = \sum_{i\in\mathcal{G}} \Phi_i(\sigma_i)$ and $\Phi_\mathcal{L}(\sigma_\mathcal{L}) = \sum_{i\in\mathcal{L}} \Phi_i(\sigma_i)$, we get

$$\frac{\partial L}{\partial \nu_{\mathcal{G}}}(x, \sigma_{-\nu_{\mathcal{L}}}) = \frac{\partial L}{\nu_{\mathcal{G}}}(x, \sigma)\Big|_{\nu^*_{\mathcal{L}}(x, \sigma_{-\nu_{\mathcal{L}}})}$$

$$= \frac{\partial \Phi_{\mathcal{G}}}{\partial \sigma_{\mathcal{G}}}(\sigma_{\mathcal{G}}) - (C_{\mathcal{G}} P)^T \tag{30a}$$

$$\frac{\partial L}{\partial \lambda_{\mathcal{G}}}(x, \sigma_{-\nu_{\mathcal{L}}}) = \frac{\partial \Phi_{\mathcal{G}}}{\partial \lambda_{\mathcal{G}}}(\sigma_{\mathcal{G}}) - (C_{\mathcal{G}} R)^T \tag{30b}$$

$$\frac{\partial L}{\partial \lambda_{\mathcal{L}}}(x, \sigma_{-\nu_{\mathcal{L}}}) = \frac{\partial \Phi_L}{\partial \lambda_{\mathcal{L}}}(\sigma_{\mathcal{L}})\Big|_{\nu^*_{\mathcal{L}}(x, \sigma_{-\nu_{\mathcal{L}}})} - (C_{\mathcal{L}} R)^T \tag{30c}$$

$$\frac{\partial L}{\partial P}(x, \sigma_{-\nu_{\mathcal{L}}}) = \nu_{\mathcal{G}}^T C_{\mathcal{L}} + \nu^*_{\mathcal{L}}(x, \sigma_{-\nu_{\mathcal{L}}})^T C_{\mathcal{L}} \tag{30d}$$

$$\frac{\partial L}{\partial R}(x, \sigma_{-\nu_{\mathcal{L}}}) = \lambda^T C \tag{30e}$$

Therefore the only apparent differences between (24) and (30) are in (30c) and (30d) where the evaluation of $\nu_{\mathcal{L}} = \nu^*_{\mathcal{L}}(x, \sigma_{-\nu_{\mathcal{L}}})$ may affect the partial derivative. However, since $\nu^*_{\mathcal{L}}(x, \sigma_{-\nu_{\mathcal{L}}})$ satisfy (29), it follows that (23) and (28) are equivalent. □

We now present our main convergence results. Let E be the set of equilibrium points of (23)

$$E := \left\{(x, \sigma) : \frac{\partial L}{\partial x}(x, \sigma) = 0,\ \frac{\partial L}{\partial \sigma}(x, \sigma) = 0\right\},$$

which by Theorem 4 is the set of optimal solutions to FP-OLC.

Theorem 6. (Global Convergence). The set E of equilibrium points of the partial primal dual algorithm (23) is globally asymptotically stable. Furthermore, each individual trajectory converges to a unique point within E that is optimal with respect to the FP-OLC problem.

Proof. By Lemma 5, we know that partial primal-dual dynamics (23) can be interpreted as a complete prima-dual gradient law of the reduced Lagrangian (27). Therefore, following (Feijer and Paganini, 2010) we consider the candidate Lyapunov function

$$U(x, \sigma_{-\nu_{\mathcal{L}}}) = \frac{1}{2}(x - x^*)^T X^{-1}(x - x^*) + \frac{1}{2}(\sigma_{-\nu_{\mathcal{L}}} - \sigma^*_{-\nu_{\mathcal{L}}})^T Z^{-1}(\sigma_{-\nu_{\mathcal{L}}} - \sigma^*_{-\nu_{\mathcal{L}}})$$

where $(x^*, \sigma^*_{-\nu_{\mathcal{L}}})$ is *any* equilibrium point of (23). Then it follows from Lemma 5 that

$$\dot{U}(x, \sigma_{-\nu_{\mathcal{L}}}) = \frac{\partial L}{\partial x}(x, \sigma_{-\nu_{\mathcal{L}}})(x^* - x) + \frac{\partial L}{\partial \sigma_{-\nu_{\mathcal{L}}}}(x, \sigma_{-\nu_{\mathcal{L}}})(\sigma_{-\nu_{\mathcal{L}}} - \sigma^*_{-\nu_{\mathcal{L}}}) \tag{31}$$

$$\begin{aligned} &\leq L(x^*, \sigma_{-\nu_{\mathcal{L}}}) - L(x, \sigma_{-\nu_{\mathcal{L}}}) + L(x, \sigma_{-\nu_{\mathcal{L}}}) - L(x, \sigma^*_{-\nu_{\mathcal{L}}}) \\ &= L(x^*, \sigma_{-\nu_{\mathcal{L}}}) - L(x, \sigma^*_{-\nu_{\mathcal{L}}}) \\ &= \underbrace{L(x^*, \sigma_{-\nu_{\mathcal{L}}}) - L(x^*, \sigma^*_{-\nu_{\mathcal{L}}})}_{\leq 0} + \underbrace{L(x^*, \sigma^*_{-\nu_{\mathcal{L}}}) - L(x, \sigma^*_{-\nu_{\mathcal{L}}})}_{\leq 0} \end{aligned} \tag{32}$$

where the first step follows from (28), the second from convexity (strict concavity) of $L(x, \sigma_{-\nu_{\mathcal{L}}})$ with respect to x ($\sigma_{-\nu_{\mathcal{L}}}$) and the last step from the fact that x^* ($\sigma^*_{-\nu_{\mathcal{L}}}$) is a minimizer (maximizer) of $L(x, \sigma_{-\nu_{\mathcal{L}}})$. Therefore, since $U(x, \sigma_{-\nu_{\mathcal{L}}})$ is radially unbounded, Lassale's Invariance Principle (Khalil, 2002) asserts that the trajectories $(x(t), \sigma_{-\nu_{\mathcal{L}}}(t))$ converge to the largest invariance set within $\{\dot{U}(x, \sigma_{-\nu_{\mathcal{L}}}) \equiv 0\}$. This implies that the trajectories $(x(t), \sigma(t))$ of (23) must converge to the largest invariant set

$$M \subseteq \{(x, \sigma) : \nu_{\mathcal{L}} = \nu^*_{\mathcal{L}}(x, \sigma_{-\nu_{\mathcal{L}}}), \dot{U}(x, \sigma_{-\nu_{\mathcal{L}}}) \equiv 0\}.$$

We now characterize M. Notice that in order to have $\dot{U} \equiv 0$, then both terms in (32) must be zero. In particular, since

$$L(x^*, \sigma_{-\nu_{\mathcal{L}}}(t)) - L(x^*, \sigma^*_{-\nu_{\mathcal{L}}}) \equiv 0.$$

and $L(x, \sigma_{-\nu_{\mathcal{L}}})$ is strictly concave on $\sigma_{-\nu_{\mathcal{L}}}$, we must have $\sigma_{-\nu_{\mathcal{L}}}(t) \equiv \sigma^*_{-\nu_{\mathcal{L}}}$ by uniqueness of the optimal solution, or equivalently

$$\nu_{\mathcal{G}}(t) \equiv 0 \text{ and } \lambda(t) \equiv \mathbf{1}\lambda^*. \tag{33}$$

Then by taking the derivative of (33) with respect to time we obtain $\dot{\nu}_{\mathcal{G}} \equiv 0$ and $\dot{\lambda} \equiv 0$.

We will now show that

$$\nu^*_L(x(t), \sigma_{-\nu_{\mathcal{L}}}(t)) \equiv \nu^*_L = 0. \tag{34}$$

Since $\sigma_{-\nu_{\mathcal{L}}}(t) \equiv \sigma^*_{-\nu_{\mathcal{L}}}$, we must have by (31)

$$0 = \frac{\partial L}{\partial x}(x, \sigma^*_{-\nu_{\mathcal{L}}})(x - x^*) \tag{35a}$$

$$= \frac{\partial L}{\partial P}(x, \sigma^*_{-\nu_{\mathcal{L}}})(P - P^*) + \frac{\partial L}{\partial R}(x, \sigma^*_{-\nu_{\mathcal{L}}})(R - R^*) \tag{35b}$$

$$= [\nu^{*T}_{\mathcal{G}} \ \ \nu^*_{\mathcal{L}}(x, \sigma_{-\nu_{\mathcal{L}}})^T] C(P - P^*) + \lambda^{*T} C(R - R^*) \tag{35c}$$

$$= \nu^*_{\mathcal{L}}(x, \sigma_{-\nu_{\mathcal{L}}})^T C_{\mathcal{L}}(P - P^*) \tag{35d}$$

$$= \left(\frac{\partial \Phi_{\mathcal{L}}}{\partial \nu_{\mathcal{L}}}(\sigma_{-\nu_{\mathcal{L}}}, \nu^*_{\mathcal{L}}(x, \sigma_{-\nu_{\mathcal{L}}})) - \frac{\partial \Phi_{\mathcal{L}}}{\partial \nu_{\mathcal{L}}}(\sigma^*_{-\nu_{\mathcal{L}}}, 0)\right) \times (\nu^*_{\mathcal{L}}(x, \sigma_{-\nu_{\mathcal{L}}}) - \nu^*_{\mathcal{L}}(x^*, \sigma^*_{-\nu_{\mathcal{L}}})) \tag{35e}$$

$$\leq 0. \tag{35f}$$

Step (35b) follows from definition of x and $L(x, \sigma_{-\nu_{\mathcal{L}}})$, (35c) from (30d) and (30e), (35d) from (33) and the fact that $C^T \mathbf{1} = 0$, (35e) follows from (29) and the fact that $\nu^*_{\mathcal{L}}(x^*, \sigma^*_{-\nu_{\mathcal{L}}}) = 0$, and (35f) follows from the monotonicity of the gradient of concave functions.

Therefore, since $\Phi_{\mathcal{L}}(\sigma_{\mathcal{L}}) = \sum_{i \in \mathcal{L}} \Phi_i(\sigma_i)$ is strictly concave, equality only holds when equation (34) holds. It also follows from (33)-(34) and (23d)-(23e) that $\dot{x} \equiv 0$. Thus, we have that $M \subseteq E$.

Now consider any trajectory of the power flows $P(t)$. Then it follows that

$$P(t) - P(0) = \int_0^t \dot{P}(s) ds = \int_0^t BC^T \nu(s) ds = BC^T \int_0^t \nu(s) ds = BC^T \theta(t).$$

where $\theta(t)$ is defined to be $\int_0^t \nu(s) ds$. Thus it follows that $P(t)$ is in an affine set characterized by the initial condition and C, i.e. $P(t) \in P(0) + \text{span}[BC^T]$, and therefore

$$(x, \sigma)(t) \in F_{P(0)} := \{(x, \sigma) : P = P(0) + BC^T \theta,\ \theta \in \mathbb{R}^{|\mathcal{N}|}\}.$$

Finally since we also know that $(x, \sigma)(t) \to M$, then we must have

$$(x, \sigma)(t) \to F_{P(0)} \cap M.$$

Now take any $(x, \sigma) \in M \cap F_{P(0)}$. Since, $M \subseteq E$, Theorem 4 implies that (x, σ) must be a primal-dual optimal solution to FP-OLC with $\nu = 0$, $\lambda = \lambda^* \mathbf{1}$ and

$$CP = P^m - d(\lambda^* \mathbf{1}). \tag{36}$$

Moreover, by definition of $F_{P(0)}$, $P = P(0) + BC^T\theta$ and we obtain

$$C(P(0) + BC^T\theta) = P^m - d^* \iff \tag{37}$$
$$CBC^T\theta = P^m - d^* - CP(0). \tag{38}$$

Notice that except for θ the terms of (38) are fixed. The matrix CBC^T is a Laplacian matrix with null space given by $\ker(CBC^T) = \operatorname{span}(\mathbf{1})$. Thus, given $\theta^1 \neq \theta^2$ both satisfying (38) we must have $(\theta^1 - \theta^2) \in \operatorname{span}(\mathbf{1})$.

Now let $P^1 = P(0) + BC^T\theta^1$ and $P^2 = P(0) + BC^T\theta^2$. Then

$$\begin{aligned} P^1 - P^2 &= (P(0) + BC^T\theta^1) - (P(0) + BC^T\theta^1) \\ &= BC^T(\theta^1 - \theta^2) = 0. \end{aligned}$$

Therefore, there is a unique vector P such that $(x, \sigma) \in M \cap F_{P(0)}$. A similar argument also shows that there is also a unique R such that $(x, \sigma) \in M \cap F_{P(0)}$.

Therefore the set $M \cap F_{P(0)}$ is a singleton to which $(x, \sigma)(t)$ converges.

$\square$

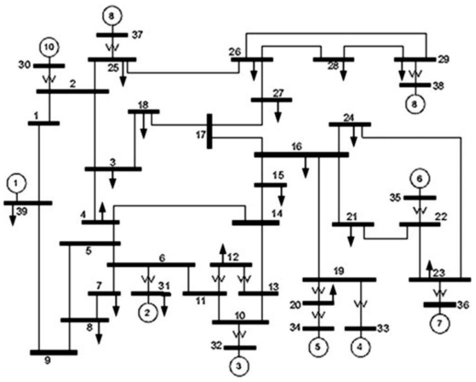

Fig. 1. IEEE 39 bus system: New England

6. NUMERICAL ILLUSTRATIONS

We now illustrate the behavior of our control scheme and compare it with the one previously proposed in (Zhao et al., 2013). We consider the widely used IEEE 39 bus system, shown in Figure 5, to test our schemes. The network parameters as well as the stationary starting point were obtained from the Power System Toolbox (Chow and Cheung, 1992) data set.

Each bus is assumed to have a controllable load with $\mathcal{D}_i = [-d_{\max}, d_{\max}]$, with $d_{\max} = 1$p.u. on a 100MVA base and disutility function

$$\begin{aligned} c_i(d_i) &= \int_0^{d_i} \tan\left(\frac{\pi}{2d_{\max}} s\right) ds \\ &= -\frac{2d_{\max}}{\pi} \ln\left(\left|\cos\left(\frac{\pi}{2d_{\max}} d_i\right)\right|\right). \end{aligned}$$

Thus, $d_i(\sigma_i) = c_i'^{-1}(\omega_i + \lambda_i) = \frac{2d_{\max}}{\pi} \arctan(\omega_i + \lambda_i)$. See Figure 2 for an illustration of both $c_i(d_i)$ and $d_i(\sigma_i)$.

Throughout the simulations we assume that the aggregate generator damping and load frequency sensitivity parameter $D_i = 0.1$ $\forall i \in \mathcal{N}$ and use $\alpha_{ij} = 2$ $\forall ij \in \mathcal{E}$. The value of these parameters does not affect convergence, but

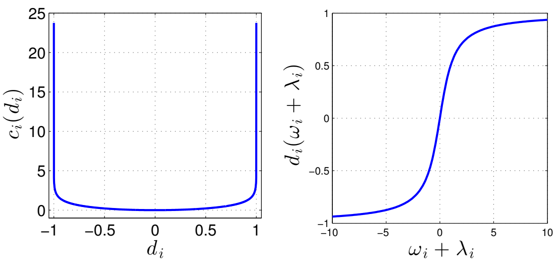

Fig. 2. Disutility $c_i(d_i)$ and load function $d_i(\omega_i + \lambda_i)$

in general will affect the convergence rate. We simulate the OLC-system proposed in (Zhao et al., 2013) as well as the FP-OLC-system (25)-(26), after introducing a perturbation at bus 1 of $P_1^m = -.5$p.u.. Figures 3 and 4 shows the evolution of the bus frequencies for the OLC and FP-OLC systems. It can be seen that while the OLC load controllers fail to recover the nominal frequency, the FP-OLC controllers can jointly rebalance the power as well as recovering the nominal frequency.

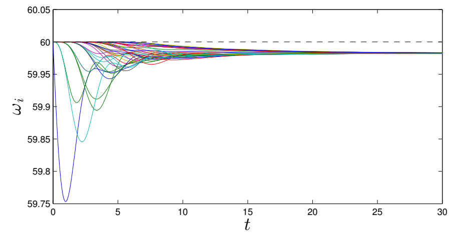

Fig. 3. Frequency evolution using OLC controllers of (Zhao et al., 2013)

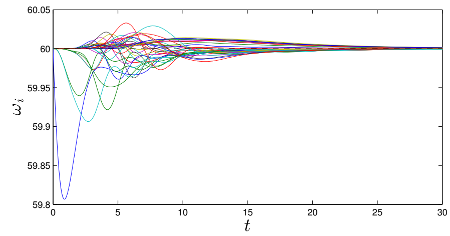

Fig. 4. Frequency evolution using FP-OLC controllers

Finally, we evaluate the "social" cost that the loads must incur in order to resynchronize the system. To compute this cost we vary the perturbation P_1^m between -10 and 10 and compute for each point the cost difference

$$\Delta C(P_1^m) := \sum_{i \in \mathcal{N}} c_i(d_i^{*,\text{FP-OLC}}(P_1^m)) - c_i(d_i^{*,\text{OLC}}(P_1^m))$$

where $d^{*,\text{OLC}}(P_1^m)$ and $d^{*,\text{FP-OLC}}(P_1^m)$ are the optimal solutions to OLC and FP-OLC respectively when the size of the perturbation is P_1^m. Similarly, we can denote $\omega_i^*(P_1^m)$ as the value of the optimal frequency when OLC is used. Figure 5 the cost $\Delta C(P_1^m)$ as a function of $\omega^*(P_1^m)$ and show how there is an additional "social" cost associated with maintaining the frequency to its nominal value.

7. CONCLUDING REMARKS

This paper studies the problem of restoring the power balance of a power network by dynamically adapting the

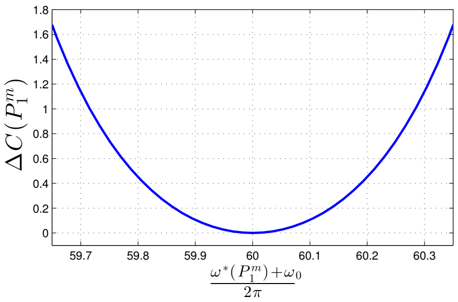

Fig. 5. Resynchronization cost

loads. We show that provided *local* communication is allowed among different buses, it is possible to rebalance the power mismatch without incurring in a frequency error. We show that our distributed solution converges for every initial condition and provide numerical simulations that verify our findings.

As future work we are interested in adapting our scheme to include the power scheduling constraints on inter area tie lines as well as studying the compatibility of our scheme with primary the primary control available on the generation side.

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